JHEP HYPER VERSION

RECEIVED: *November 1, 1999,* ACCEPTED: *January 28, 2000*

Correlation functions for chiral primaries in D=6 Supergravity on $AdS_3 \times S^3$

Mihail Mihailescu

Brown University
Providence, RI 02912
E-mail: mm@het.brown.edu

ABSTRACT: Six dimensional supergravities on $ADS_3 \times S^3$ present interest due to the role they play in the AdS/CFT correspondence. The correspondence in this case states the equivalence between supergravity on the given background and a still unknown conformal field theory. The conformal field theory in question is expected to appear by deforming the S^N orbifold conformal field theory on T^4 in a way which preserves the superconformal symmetry. The purpose of this paper is to compute the first nontrivial corrections to the equations of motion for the chiral primary fields coming from supergravity. Using the methods already developed which involve nontrivial redefinitions of fields, we compute three-point correlation functions for scalar chiral primaries and notice similarities between their expressions and those obtained in the orbifold conformal field theory.

KEYWORDS: D-branes, Black Holes in String Theory, Conformal and W Symmetry, Supergravity Models.

JHEP02(2000)007

Contents

1. Introduction

The correspondence [1] between CFT and compactified supergravity on AdS spaces is a topic under vigorous current investigation. The first check of the correspondence was done by comparison of the spectra of chiral primaries in both theories and was followed by comparison of three point interactions. Such comparison involve computation of three point Greens functions in Yang-Mills theory (without interactions) and then, the evaluation of the analogue three point vertices in compactified supergravity [2, 3, 4, 5]. An agreement of the two (even for the range of couplings where it should not hold) is due to some rather nontrivial nonrenormalization theorems.

One of the most interesting case of Maldacena's conjecture is the one of IIB string theory compactified on a $K3$ or T^4 manifold [6]. Here, a distribution of five branes and one-branes gives a black string whose near horizon geometry is $AdS_3 \times S^3 \times M^4$. Further compactification on S^1 produces a five-dimensional black hole. Its entropy is nicely accounted [7] by a conformal field theory with target space $S^N(T^4)$ or $S^N(K3)$. This orbifold conformal field theory was recently studied in a number of works [8, 9, 10]. In [11], a suggestion that it originates in a non-commutative space-time was given, while the nonlinear aspects of this conformal field theory were discussed in [12]. In particular, a construction of chiral primary operators was explicitely given and their operator products evaluated. This gave specific answers for the three point correlation functions. The orbifold framework allowed for a discussion of the corresponding spectrum generating algebra which seems to govern the dynamics of the theory. It is clearly relevant to perform the analogous calculations at the supergravity level. In this case, the relevant $6D$ supergravity involves nontrivial multiplets which have their origin in compactification on $K3$ or T^4. Furthermore, the precise correspondence is actualy not known, in particular it is likely that a deformed conformal field theory might ultimately be relevant. Even though some of the deformations have been mentioned, explicite calculations were not done yet.

In this paper, detailed computations of cubic interactions among chiral primaries of $6D$ supergravity will be presented. The computation involves the standard elements of previous works, namely nonlinear field redefinitions, which then leads to three point functions of chiral operators. The results presented at the end of section 4 have some similarities in form with the previous examples done in the literature (which all have a universal form in dependence on the chiral quantum numbers l) [13, 14]. In our case, we observe for one sequence of operators a deviation from the universal form and a particular quadratic dependence on the quantum numbers l. It is a very interesting question to ask if a deformation of the free orbifold CFT would give an identical result.

The paper is organized as follows: in section 2 we give following [15, 16] the basic notation and definition of $6D$ supergravity with the discussion of chiral primaries. In section 3 the action is expanded to cubic order and the nonlinear field redefinitions are determined. Section 4 discusses the corresponding correlation functions. Some comments on future work are made in the conclusions.

2. Supergravity in $D = 6$

In this section, we review following [15, 16] the equations of motion of $D = 6$ supergravity of interest for us in determining the scalar chiral primaries. We are interested in the equation of motion for metric g_{MN}, 2-forms B^i_{MN}, B^r_{MN}, $i = 1, \ldots, 5$, $r = 1, \ldots, n$ and scalars. The scalars are packaged in a $\mathrm{SO}(5, n)$ matrix $V_I{}^J$, $I, J = 1, \ldots, 5 + n$ giving a parametrization of the coset $\mathrm{SO}(5, n)/(\mathrm{SO}(5) \times \mathrm{SO}(n))$. The integer n depends on the type of supergravity we study in $6D$: for $N = 4b$ supergravity, coming from a compactification of type-IIB supergravity on $K3$, $n = 21$ and for $N = 8$ supergravity, coming from compatification of the same $10D$ theory on T^4, $n = 5$. As long as we work with the metric, the scalars and two-forms only, we can use n unspecified. We will use the following notation for indices: $M, N = 0, \ldots, 5$ are $D = 6$ coordinates, $\mu, \nu = 0, \ldots, 2$ are AdS_3 coordinates and $a, b = 1, \ldots, 3$ are S^3 coordinates. We will also use $G^I = dB^I$.

We construct the following quantities which will be used in writing the equations of motion in a compact form:

$$
\begin{aligned}
dVV^{-1} &= \begin{pmatrix} Q^{ij} & \sqrt{2}P^{is} \\ \sqrt{2}P^{jr} & Q^{rs} \end{pmatrix}, \\
D_M P^{ir}_N &= \nabla_M P^{ir}_N - Q^{ij}_M P^{jr}_N - Q^{rs}_M P^{is}_N \, , \\
H^i &= G^I V_I{}^i \, , \qquad H^r = G^I V_I{}^r \, .
\end{aligned} \tag{2.1}
$$

Then, the $D = 6$ supergravity equations of motion are:

$$
\begin{aligned}
& R_{MN} = H^i_{MPQ} H^{i\,PQ}_N + H^r_{MPQ} H^{r\,PQ}_N + 2P^{ir}_M P^{ir}_N \, , \\
& D^M P^{ir}_M - \frac{\sqrt{2}}{3} H^i_{MNP} H^{r\,MNP} = 0 \, , \\
& *H^i = H^i \, , \qquad *H^r = -H^r \, .
\end{aligned} \tag{2.2}
$$

We consider a vacuum solution which is maximally supersymmetric in $D = 6$ and has the geometry of $AdS_3 \times S^3$ with radius and cosmological constant equal to one. It consists in the corresponding metric $\bar{g}_{MN}$ of the direct product of the two spaces and the following ansatz for the forms and scalars: $\bar{V} = I_{5+n}$, $\bar{G}^5_{\mu\nu\rho} = \epsilon_{\mu\nu\rho}$, $\bar{G}^5_{abc} = \epsilon_{abc}$ and all the other $\bar{G}$'s set to zero. In the previous equations $\epsilon_{\mu\nu\rho}$ and ϵ_{abc} are the volume forms on AdS_3 and S^3 respectively such that, $\epsilon_{\mu\nu\rho abc} = \epsilon_{\mu\nu\rho}\ \epsilon_{abc}$ is the volume form in six dimensions. We will recover later by scaling arguments the radius dependence.

3. Scalar chiral primaries

We parameterize the scalar manifold by introducing scalar fields ϕ^{ir} representing a $5 \times n$ matrix Φ which gives the fluctuations of V away from identity. We write all the fields in second order in fluctuations:

$$
\begin{aligned}
g_{MN} &= \bar{g}_{MN} + h_{MN} \, , \\
G^I &= \bar{G}^I + g^I \, , \\
V &= \begin{pmatrix} I_5 + \frac{1}{2}\Phi\Phi^T & \Phi \\ \Phi^T & I_n + \frac{1}{2}\Phi^T\Phi \end{pmatrix},
\end{aligned} \tag{3.1}
$$

Then, we write the quantities defined in (2.1) in this order and we obtain:

$$
\begin{aligned}
Q^{ij} &= \frac{1}{2}(\phi^{ir} d\phi^{jr} - \phi^{jr} d\phi^{ir}) \, , \\
Q^{rs} &= \frac{1}{2}(\phi^{ir} d\phi^{is} - \phi^{is} d\phi^{ir}) \, , \\
P^{ir} &= \frac{1}{\sqrt{2}} d\phi^{ir} \, , \\
H^i &= g^i + g^r \phi^{ir} + \frac{1}{2}\bar{G}^5 \phi^{5r} \phi^{ir} \, , \qquad i \neq 5 \, , \\
H^5 &= \bar{G}^5 + g^5 + g^r \phi^{5r} + \frac{1}{2}\bar{G}^5 \phi^{5r} \phi^{5r} \, , \\
H^r &= g^r + \bar{G}^5 \phi^{5r} + g^i \phi^{ir} \, ,
\end{aligned} \tag{3.2}
$$

We focus now, on scalar chiral primaries only. As it was already shown in [16], they come from two sources: for $r = 1, \ldots, n$ from the equation of motion for the two-forms B^r and scalars ϕ^{5r}, and for the last one from the equation of graviton and the two-form B^5. We also use the parametrization of the two-forms given in [16]. The forms are decomposed in spherical harmonics of S^3 and we use the gauge invariance and the allowed diffeomorphism invariance to reduce the fluctuations to physical ones:

$$\begin{aligned} & h_{ab} = N\bar{g}_{ab}\,, \qquad h_{\mu a} = 0\,, \qquad h_{\mu\nu} = M\bar{g}_{\mu\nu} + h'_{(\mu\nu)}\,, \\ & g^I_{MNP} = 3\partial_{[M} b^I_{NP]}\,, \\ & b^I_{\mu\nu} = \epsilon_{\mu\nu\rho} X^{I\ \rho}\,, \qquad b^I_{\mu a} = 0\,, \qquad b^I_{abc} = \epsilon_{abc}\partial^c U^I\,, \end{aligned} \tag{3.3}$$

where $h'_{(\mu\nu)}$ is the traceless part of the graviton fluctuation on AdS_3.

Let us start with the first category of chiral primaries and solve the linearized equations leading to those (see appendix A):

$$(\nabla^\mu\nabla_\mu + \nabla^a\nabla_a)\phi^{5r} + 4(\nabla^\mu\nabla_\mu - \nabla^a\nabla_a)U^r = 4\nabla^\mu(\nabla_\mu U^r - X^r_\mu) + Q^r_1\,, \tag{3.4}$$

from the equation of motion for scalars ϕ^{5r},

$$\nabla_a X^r_\mu - \nabla_a\nabla^\mu U^r = Q^r_{2\ a\mu}\,, \tag{3.5}$$

and

$$(\nabla^\mu\nabla_\mu + \nabla^a\nabla_a)U^r + 2\phi^{5r} = \nabla^\mu(\nabla_\mu U^r - X^r_\mu) + Q^r_3, \tag{3.6}$$

from the $\mu\nu a$ and abc component of the antiself-duality equation for the two-forms B^r. $Q^r_{1,2,3}$ are the second order corrections to the equations and they are determined in the next section. We expand the scalars ϕ^{5r} and those used to parametrize the fluctuations of two-forms in the harmonic basis of S^3 and we get after solving the constraint (3.5), a system of equations, (3.4) and (3.6), which has to be diagonalized. We arrive at the following definitions: for the chiral primaries $s^{r\,I} = \frac{1}{4(l+1)}(\phi^{5r\,I} + 2(l+2)U^{r\,I})$ and for the other scalar $t^{r\,I} = \frac{1}{4(l+1)}(\phi^{5r\,I} - 2lU^{r\,I})$, where I refers from now on to the harmonic index on S^3 and l refers to the integer entering the Casimir of that representation: $l(l+2)$. We also have, if we put to zero all the fields except the chiral primaries: $\phi^{5r\,I} = 2ls^{r\,I}$, $U^{r\,I} = s^{r\,I}$.

For the second category, those coming from the graviton and one of the two-forms, we have the following equations at the linearized level:

$$-\frac{1}{2}(\nabla^\mu\nabla_\mu + \nabla^a\nabla_a)N - \frac{1}{6}\nabla^a\nabla_a(3M+N) = -4N + 4\nabla^a\nabla_a U^5 + Q^\sigma_1\,, \tag{3.7}$$

and

$$(3M+N)_{;(ab)} = Q^\sigma_{2\,(ab)}\,, \tag{3.8}$$

coming from the ab component of the Einstein equation, (3.7) is the part proportional to $\bar{g}_{ab}$ and (3.8) is the symmetric traceless part (ab);

$$\nabla_a X^5_\mu + \nabla_a \nabla^\mu U^5 + Q^\sigma_{3\,a\mu} = 0\,, \tag{3.9}$$

and

$$(\nabla^\mu\nabla_\mu + \nabla^a\nabla_a)U^5 - 2N = \nabla^\mu(\nabla_\mu U^5 + X^5_\mu) + Q^\sigma_4 \tag{3.10}$$

from the $\mu\nu a$ and abc component of the self-duality equation for the two-form B^5. $Q^\sigma_{1,2,3,4}$ are the second order corrections of the equations. We solve again the constraints (3.8) and (3.9), and we obtain a system of equations (3.7) and (3.10) which is diagonalized by the following expressions: the chiral primaries given by $\sigma^I = \frac{1}{4(l+1)}(-N^I + 2(l+2)U^{5\,I})$, and another nonchiral primary scalar $\tau^I = \frac{1}{4(l+1)}(N^I + 2lU^{5\,I})$. If we set to zero all fields, but the chiral primaries we obtain the following expressions: $N^I = -2l\sigma^I$, $U^{5\,I} = \sigma^I$. We solve also the linearized μa part of the graviton equation and we obtain that $h'^I_{(\mu\nu)} = -\frac{4}{l+1}\nabla_{(\mu}\nabla_{\nu)}\sigma^I$. We note that in the above equations, namely at the linearized level, $h'^I_{(\mu\nu)}$ does not enter. It apears only in the second order variation of equations of motion (in Q's), for our purpouses being enough to compute only its first order expresssion in term of chiral primaries.

4. The action in AdS_3 for chiral primaries

We compute in this section the second order corrections to the equations of motion for chiral primaries. We notice that some of the second order terms involve derivatives on AdS_3 of chiral primaries. Because a consistent field theory believed to exist for the chiral primaries requires the absence of these terms, we will redefine the chiral primaries in such a way that the new terms cancel the problematic terms. These redefinitions of chiral fields are also consistent to the fact that we named the chiral primaries after solving linearized equations only. It will be intersting to test if using the full nonlinear expression for the supersymmetry algebra and the conditions for a field to be chiral we could confirm our redefinitions of fields.

Let us write the expressions for the corrections using partial results computed in appendix A. We start with the corrections given in Q^r's:

$$\begin{aligned}
Q^r_1 &= \nabla_\mu(h^{\mu\nu}\nabla_\nu\phi^{5r}) + \nabla_a(h^{ab}\nabla_b\phi^{5r}) - \frac{1}{2}\nabla^P(h_M^{\ M})\nabla_P\phi^{5r} + \\
&\quad + 4\Big(h_\mu^{\ \mu}\nabla_\nu\nabla^\nu U^r - h_a^{\ a}\nabla_b\nabla^b U^r + 2\phi^{5r}(\nabla_\nu\nabla^\nu U^5 + \nabla_b\nabla^b U^5 - 2N)\Big) + \\
&\quad + 4\Big(\nabla_\nu\nabla^\nu U^5\nabla_\rho\nabla^\rho U^r + \nabla_b\nabla^b U^5\nabla_c\nabla^c U^r + 2\nabla_b\nabla_\nu U^5\nabla^b\nabla^\nu U^r\Big)\,, \\
Q^r_{2\ a\mu} &= 2\nabla_a\nabla_\mu U^5\phi^{5r} + \nabla_a\nabla_\mu U^r\left(h_\mu^{\ \mu} - \frac{1}{2}h_M^{\ M}\right) + h_a^{\ b}\nabla_b\nabla_\mu U^r - h_\mu^{\ \nu}\nabla_a\nabla_\nu U^r\,, \\
Q^r_3 &= (\nabla_\nu\nabla^\nu U^5 - \nabla_b\nabla^b U^5)\phi^{5r} - \left(\frac{1}{2}h_M^{\ M} - h_a^{\ a}\right)(\nabla_b\nabla^b U^r + \phi^{5r})\,.
\end{aligned} \tag{4.1}$$

Next, we use the expansion of fields in sperical harmonics and compute the correpronding corrections for each I. For definitions and derivations for spherical harmonics see appendix B. Here, we only repeat a few definitions: $\Delta = l(l+2)$ are up to a minus sign the eigenvalues of the sphere laplacean, $\delta = l(l-2)$ are related to the masses of the chiral primaries, $C(I)$ is the normalization constant of the I-th spherical harmonic and is obtained from the integral over S^3 of the square of it, and $C(I, I_1, I_2)$ is the integral of the corresponding product of three spherical harmonics.

We solve first the constraint (3.5) and from it we obtain:

$$\begin{aligned}\nabla_\mu U^{r\ I} - X_\mu^{r\ I} = \frac{C(I,I_1,I_2)}{\Delta C(I)}\Big(&-2l_1(\Delta+\Delta_2-\Delta_1)s^{r\ I_1}\nabla_\mu\sigma^{I_2} + \\ &+(\Delta+\Delta_1-\Delta_2)\Big(-l_2\nabla_\mu s^{r\ I_1}\sigma^{I_2} + \frac{1}{2}\nabla_\nu s^{r\ I_1}h_\mu^{I_2\ \nu}\Big)\Big)\,.\end{aligned} \tag{4.2}$$

We expand the equations (3.6) and (3.4) in spherical harmonics and for each I we sum them with the followig weights: (3.6) with $(l+2)/2(l+1)$ and (3.4) with $1/4(l+1)$. In the end, we obtain the following equation for $s^{r\,I}$ chiral primaries including second order corrections:

$$\begin{aligned}(\nabla_\nu\nabla^\nu - l(l-2))s^{r\,I} = \frac{1}{4(l+1)}\Big(&2(l+2)\nabla^\mu(\nabla_\mu U^{r\,I} - X_\mu^{r\,I}) + \\ &+\frac{C(I,I_1,I_2)}{C(I)}\Big(\nabla_\mu(\nabla_\nu\phi^{5r\,I_1}h^{I_2\,\mu\nu}) - \Delta_1\phi^{5r\,I_1}N^{I_2} - \\ &-\nabla_\mu\phi^{5r\,I_1}\nabla^\mu N^{I_2} + 4\Big((-\nabla_\nu\nabla^\nu + 3\Delta_1)U^{r\,I_1}N^{I_2} - \\ &-4\phi^{5r\,I_1}N^{I_2} + (-2\phi^{5r\,I_1} + \Delta_1 U^{r\,I_1})\Delta_2 U^{5\,I_2} + \\ &+(\nabla_\nu\nabla^\nu U^{r\,I_1} + 2\phi^{5r\,I_1})\nabla_\nu\nabla^\nu U^{5\,I_2} + \\ &+(\Delta_1+\Delta_2-\Delta)\nabla_\nu U^{r\,I_1}\nabla^\nu U^{5\,I_2}\Big) + \\ &+2l\Big(\phi^{5r\,I_1}(\nabla_\nu\nabla^\nu + \Delta_2)U^{5\,I_2} + 2(-\Delta_1 U^{r\,I_1} + \phi^{5r\,I_1})N^{I_2}\Big)\Big)\Big).\end{aligned} \tag{4.3}$$

We use now the expression for the fields ϕ^{5r}, U^r, U^5, N, $h_{\mu\nu}$,... in term of chiral primaries $s^{r\,I}$ and as it is expected, we obtain equations which have in the right hand side terms involving derivatives $\nabla_\nu s^{r\,I}$. These terms suggest that the definition of chiral primary fields is changed at the quadratic level. We remove the unwanted terms by redefinitions and we end up with the following rules which are applied in the right hand side of (4.3). We consider s, σ as generic fields which at the linearized level solve $(\nabla_\nu\nabla^\nu - l_1(l_1-2))s = 0$ and $(\nabla_\nu\nabla^\nu - l_2(l_2-2))\sigma = 0$ respectively, and the rules are valid if the corrected field solves at linearized level $(\nabla_\nu\nabla^\nu - l(l-2)l)\cdots = 0$.

$$\begin{aligned}\nabla_\nu(\nabla^\nu s\sigma) &\longrightarrow \frac{1}{2}(\delta+\delta_1-\delta_2)s\sigma\,, \\ \nabla^\nu s\nabla_\nu\sigma &\longrightarrow \frac{1}{2}(\delta-\delta_1-\delta_2)s\sigma\,,\end{aligned}$$

$$\nabla_\mu(\nabla_\nu s \nabla^\mu \nabla^\nu \sigma) \longrightarrow \frac{1}{4}(\delta+\delta_2-\delta_1)(\delta-\delta_1-\delta_2)s\sigma\,. \tag{4.4}$$

We mention here, that the above rules are valid only for second order corrections and they have to be changed or replaced for the higher orders. Plugging in (4.3) the expression for $X_\mu^{r\ I}$ obtained from equation (4.3) and using then (4.4) to remove the derivatives terms we will obtain the following equations:

$$(\nabla_\nu\nabla^\nu - l(l-2))s^{r\,I} = \frac{C(I,I_1,I_2)}{C(I)}\frac{\alpha\alpha_1\alpha_2(\Sigma-2)\Sigma(\Sigma+2)}{4l(l+1)(l_2+1)}s^{r\,I_1}\sigma^{I_2}\,, \tag{4.5}$$

where $\alpha = l_1+l_2-l$, $\alpha_1 = l+l_2-l_1$, $\alpha_2 = l+l_1-l_2$ and $\Sigma = l_1+l_2+l$.

We will work now, in a similar fashion the quadratic corrections to the equations of motion for the chiral primary coming from the metric and one of the self-dual forms, σ. There are two kinds of quadratic corrections: those which are quadratic in $s^{r\,I}$ and those which are quadratic in σ^I. We work each correction separately. We start with those coming from the first category (we will denote both in the same way, Q's):

$$\begin{aligned}
Q_1^\sigma &= 4\nabla_b\nabla^b U^r\phi^{5r} + 2\phi^{5r}\phi^{5r} + 2(\nabla_b\nabla^b U^r + \phi^{5r})^2 + \frac{2}{3}(\nabla_b\nabla_\nu U^r)^2 + \frac{1}{3}(\nabla_b\phi^{5r})^2\,,\\
Q^\sigma_{2\ (ab)} &= 8\nabla_{(a}\nabla^\nu U^r\nabla_{b)}\nabla_\nu U^r - 2\nabla_{(a}\phi^{5r}\nabla_{b)}\phi^{5r}\,,\\
Q^\sigma_{3\ a\mu} &= 2\nabla_a\nabla_\mu U^r\phi^{5r}\,,\\
Q_4^\sigma &= (\nabla_\nu\nabla^\nu - \nabla_b\nabla^b)U^r\phi^{5r} - \frac{1}{2}(3M+N)\,.
\end{aligned} \tag{4.6}$$

We compute from (3.8) and (3.9) the quadratic corrections to M^I and $X_\mu^{I\ 5}$ and plug them back in the system of equations formed from (3.7) and (3.10). We sum then the two equations with the appropriate factors: $-1/4(l+1)$ for (3.7) and $(l+2)/2(l+1)$ for (3.9), and the result is the linearized equation for σ^I and its quadratic corrections. Expressing all the fields in term of $s^{r\,I}$ and σ^I we obtain for the constraints:

$$\begin{aligned}
3M^I+N^I &= \frac{3C(I,I_1,I_2)}{2\Delta(\Delta-3)C(I)}\left(\frac{1}{4}(\Delta_1+\Delta-\Delta_2)(\Delta+\Delta_2-\Delta_1)+\right.\\
&\qquad\left.+\frac{\Delta}{6}(\Delta_1+\Delta_2-\Delta)\right)\times\\
&\qquad\times 8(\nabla^\nu s^{r\,I_1}\nabla_\nu s^{r\,I_2} - l_1l_2 s^{r\,I_1}s^{r\,I_2}) + \cdots\,,\\
\nabla^\nu(X_\nu^{I\,5}+\nabla_\nu U^{I\,5}) &= \frac{-2l_2C(I,I_1,I_2)}{\Delta C(I)}(\Delta_1+\Delta-\Delta_2)\nabla^\nu(\nabla_\nu s^{r\,I_1}s^{r\,I_2}) + \cdots\,,
\end{aligned} \tag{4.7}$$

where the dots represent the other kind of corrections, due to σ^I. The final equation is:

$$
\begin{aligned}
(\nabla_\nu\nabla^\nu - l(l-2))\sigma^I = & \frac{1}{4(l+1)}\left(\left(-\frac{\Delta}{3} - l - 2\right)(3M^I + N^I) + \right. \\
& + 2(l+2)\nabla^\nu(X_\nu^{I\,5} + \nabla_\nu U^{I\,5}) + \frac{2C(I,I_1,I_2)}{C(I)}\left(-4\Delta_1 U^{r\,I_1}\phi^{5r\,I_2} + \right. \\
& + 2\phi^{5r\,I_1}\phi^{5r\,I_2} + 2(-\Delta_1 U^{r\,I_1} + \phi^{5r\,I_1})(-\Delta_2 U^{r\,I_2} + \phi^{5r\,I_2}) + \\
& + (\Delta_1 + \Delta_2 - \Delta)\left(\frac{1}{3}\nabla^\nu U^{r\,I_1}\nabla_\nu U^{r\,I_2} + \frac{1}{6}\phi^{5r\,I_1}\phi^{5r\,I_2}\right) + \\
& \left.\left. + (l+2)(\nabla_\nu\nabla^\nu + \Delta_1)U^{r\,I_1}\phi^{5r\,I_2}\right)\right) + \cdots
\end{aligned} \tag{4.8}
$$

Rewriting all the fields in term of chiral primaries and using the given rules (4.4) to remove the quadratic corrections with derivatives we obtain after symetrization in $I_{1,2}$ the following result:

$$
(\nabla_\nu\nabla^\nu - l(l-2))\sigma^I = \frac{\alpha\alpha_1\alpha_2(\Sigma-2)\Sigma(\Sigma+2)}{8(l-1)l(l+1)}\frac{C(I,I_1,I_2)}{C(I)}s^{r\,I_1}s^{r\,I_2} + \cdots \tag{4.9}
$$

We turn now to the quadratic corrections due only to σ^I:

$$
\begin{aligned}
Q_1^\sigma &= -\frac{1}{3}\nabla_b\nabla^b(N^2) - \frac{1}{2}(N_\mu h^{\mu\nu})_{;\nu} - \frac{1}{12}\nabla_b\nabla^b(h^{\mu\nu}h_{\mu\nu}) + \frac{1}{12}\nabla^b h^{\mu\nu}\nabla_b h_{\mu\nu} + \\
&\quad + \frac{1}{12}\nabla^b N\nabla_b N + 6N^2 - 8N\nabla_b\nabla^b U^5 + +2(\nabla_b\nabla^b U^5)^2 + \frac{2}{3}(\nabla_b\nabla_\nu U^5)^2\,, \\
Q_{2\,(ab)}^\sigma &= \frac{1}{2}\Big((N^2)_{;(ab)} + (h^{\mu\nu}h_{\mu\nu})_{;(ab)} - \nabla_{(a}h^{\mu\nu}\nabla_{b)}h_{\mu\nu} - \nabla_{(a}N\nabla_{b)}N\Big) + \\
&\quad + 8\nabla_{(a}\nabla_\nu U^5\nabla_{b)}\nabla^\nu U^5\,, \\
Q_{3\,a\mu}^\sigma &= -N\nabla_a\nabla_\mu U^5 - \nabla_a\nabla_\nu U^5 h_\mu^\nu\,, \\
Q_4^\sigma &= 2N\nabla_b\nabla^b U5 - \frac{11}{4}N^2 + \frac{1}{4}h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}(3M+N)\,.
\end{aligned} \tag{4.10}
$$

In the same way as before, we obtain the following corrected equations of motion after using the expressions of the $D=6$ fields in terms of the chiral primaries, of the rules (4.4) and of symmetrization in the spherical indices I_1, I_2:

$$
\begin{aligned}
(\nabla_\nu\nabla^\nu - l(l-2))\sigma^I = \cdots + & \frac{\alpha\alpha_1\alpha_2(\Sigma-2)\Sigma(\Sigma+2)}{16(l-1)l(l+1)(l_1+1)(l_2+1)} \times \\
& \times(l^2 + l_1^2 + l_2^2 - 2)\frac{C(I,I_1,I_2)}{C(I)}\sigma^{I_1}\sigma^{I_2}\,,
\end{aligned} \tag{4.11}
$$

where by $\cdots$ we represent the already computed part, namely, the r.h.s. of (4.9).

It is now possible to write the action compatible with the determined equations (4.5), (4.9) and (4.11) whithout being necessary to compute from somewhere else the quadratic terms. The $D=3$ lagrangean, up to normalization constants, is:

$$\begin{aligned} L_{\text{chiral}} &= 4l(l+1)C(I)((\partial s^{r\,I})^2 + l(l-2)(s^{r\,I})^2) + 4l(l-1)C(I)\times \\ &\times((\partial\sigma^I)^2 + l(l-2)(\sigma^I)^2) + \frac{\alpha\alpha_1\alpha_2(\Sigma-2)\Sigma(\Sigma+2)}{(l+1)}C(I,I_1,I_2)\sigma^I s^{r\,I_1}s^{r\,I_2} + \\ &+ \frac{\alpha\alpha_1\alpha_2(\Sigma-2)\Sigma(\Sigma+2)}{6(l+1)(l_2+1)(l_2+1)}(l^2+l_1^2+l_2^2-2)C(I,I_1,I_2)\sigma^I\sigma^{I_1}\sigma^{I_2}\,. \end{aligned} \tag{4.12}$$

5. Correlation functions

In this section, we derive the needed constant required in equation (4.12) from the higher dimensional theory and compute the CFT correlation functions. We start with a system of Q_1 $D1$-branes and Q_5 $D5$-branes in type-IIB string theory. We review the decoupling limit following [1, 6] and deduce from it the constant required in the equations of motion of $D=6$ supergravity. The following constants have the known definitions: g is the string coupling, α' is the inverse of the string tension, v is the volume of T^4 and $g_6 = g/v^{1/2}$ is the $D=6$ coupling constant. The $D=10$ metric in the extremal case and in the decoupling limit: $r\to 0$, $\alpha'\to 0$, such that $U\equiv r/\alpha'$ is kept fixed, gives:

$$ds^2 = \alpha'\left(\frac{U^2}{l^2}(-dt^2+dx^{5\;2}) + \frac{l^2}{U^2}dU^2 + l^2 d\Omega_3^2 + \sqrt{\frac{Q_1}{Q_5}}dx^i dx^i\right), \tag{5.1}$$

where x^i, $i=5,\ldots,9$ have 2π period, and $l^2 = g_6(Q_1Q_5)^{1/2}$.

Following [1] we require the constants to be such that the radius of the sphere to be one: $\alpha' l^2 = 1$. In this limit the volume of T^4 shrinks to zero, justifying the use of $D=6$ supergravity coming from KK reduction of the $10D$ theory and the constant appearing in the $D=6$ action is given by:

$$\frac{1}{g_6^2\alpha'^2}\longrightarrow N = Q_1\,Q_5\,. \tag{5.2}$$

From now on, we use such units that the overall constant in front of the CFT action is only N, besides l dependence. For each chiral primary field in supergravity there is a corresponding one in CFT and in our case the correspondence is: $s^r\longrightarrow O^r$ and $\sigma\longrightarrow O_0$. The nonzero two point functions are then:

$$\begin{aligned} \langle O^{r\,I}(x)\,O^{s\,I_1}(y)\rangle &= N\frac{l^2(l-1)}{2^{l-2}}\frac{\delta^{rs}\delta^{II_1}}{|x-y|^{2l}}\,, \\ \langle O_0^I(x)\,O_0^{I_1}(y)\rangle &= N\frac{l^2(l-1)^2}{2^{l-2}(l+1)}\frac{\delta^{II_1}}{|x-y|^{2l}}\,. \end{aligned} \tag{5.3}$$

The two-point functions are normalized by multiplying the fields by appropriate factors: $O^{I\,r}$ by $(2^{l-2}/Nl^2(l-1))^{1/2}$ and O_0^I by $(2^{l-2}(l+1)/Nl^2(l-1)^2)^{1/2}$. Then the non-zero constants appearing in the three-point functions are given as:

$$\langle O_0^I O^{r\,I_1} O^{s\,I_2} \rangle = \frac{\delta^{rs}}{N^{1/2}} \frac{(l_1-1)^{1/2}(l_2-1)^{1/2}}{(l+1)^{1/2}} \langle \Omega^I \Omega^{I_1} \Omega^{I_2} \rangle \,,$$
$$\langle O_0^I O_0^{I_1} O_0^{I_2} \rangle = \frac{1}{6N^{1/2}} \frac{l^2+l_1^2+l_1^2-2}{(l+1)^{1/2}(l_1+1)^{1/2}(l_2+1)^{1/2}} \langle \Omega^I \Omega^{I_1} \Omega^{I_2} \rangle, \qquad (5.4)$$

where $\langle \Omega^I \Omega^{I_1} \Omega^{I_2} \rangle$ keeps track of the SO(4) Clebsch-Gordon coefficients, coming from spherical harmonics, in the structure constants.

6. Conclusions

In this paper, we gave the calculation of three point interactions of chiral primaries in 6D supergravity. Deriving the interactions, we used nontrivial redefinitions of the fields which solved the linear equation of motion for chiral primaries. The reasons behind these redefinitions are still not clear. The justification in our case is to use them to remove unwanted derivative terms in the second order equations of motion. The appearance of the derivative terms shows in a way the lack in understanding of supergravity compactified on spheres. It is after removing the derivative terms that we deal with a standard field theory in three dimensions. It would be very interesting to find a less technical principle to justify them in all orders. A step in this direction was made in [17] for a different case than ours and for a much smaller number of fields, namely those appearing in a consistent sphere truncation of the corresponding supergravity.

In section 4, we derived the three point interactions corresponding to the expected conformal field theory in two dimensions. If we compare results presented at the end of section 4 with the three point interactions found in the free orbifold CFT [12] there are similarities, but they are different overall. The similarities show that it is still possible to expect a gravitational theory to be equivalent to the orbifold CFT, but the question is which one. On the other side the differences are expected since the two descriptions do not sit at the same point in moduli space. A relevant role in this comparison is played by the dynamical symmetry groups: on the gravity side there is a manifest SO(5) for T^4 or SO(21) for $K3$, symmetry at the lagrangean level. The backround we used, respects this symmetry and so does the expression for three point interactions derived in section 4. In the CFT only the SO(4) or SO(20) subgroup is present. At present, the dynamical mechanism by which this is enlarged is not known. To achieve an exact comparison one should perform analogous computations in a different gravitational background or equivalently have a deformation of the CFT. This interesting question is left for future studies.

JHEP02(2000)007

Acknowledgments

I am gratefull to Antal Jevicki for many helpfull disscussions and for his assistance in the preparation of this manuscript. I am also happy to acknowledge instructive discussions with Sanjaye Ramgoolam and Radu Tatar.

I gratefully acknowledge a fellowship from the Galkin Foundation.

A. First and second order variations of fields

We give in this appendix detailed formulas of first (upper index (1)) and second (upper index (2)) order variation of the $D = 6$ fields used in text.

We consider first the metric, the Christoffel symbols and the curvature tensor:

$$\begin{aligned}
g^{MN} &= \bar{g}^{MN} - h^{MN} + h^{2\ MN} + \cdots\,, \\
\Gamma^P_{MN} &= \bar{\Gamma}^P_{MN} + \Gamma^{(1)\ P}_{MN} + \Gamma^{(2)\ P}_{MN} + \cdots\,, \\
\Gamma^{(1)\ P}_{MN} &= \frac{1}{2}\left(h_{M;N}^{\ \ P} + h_{N;M}^{\ \ P} - h_{MN;}^{\ \ \ \ P}\right), \\
\Gamma^{(2)\ P}_{MN} &= \frac{1}{2} h^{PR}(h_{RM;N} + h_{RN;M} - h_{MN;R})\,, \\
R_{MN} &= \bar{R}_{MN} + \left(\Gamma^{(1)P}_{MN;P} - \Gamma^{(1)P}_{MP;N}\right) + \Big(\Gamma^{(2)P}_{MN;P} - \Gamma^{(2)P}_{MP;N} + \Gamma^{(1)\ P}_{RP}\Gamma^{(1)\ R}_{MN} - \\
&\qquad - \Gamma^{(1)\ P}_{RN}\Gamma^{(1)\ R}_{MP}\Big) + \cdots\,,
\end{aligned} \tag{A.1}$$

We are intersted only when $M = a, N = b$ and when the metric fluctuation are parametrized like in (3.3) and the results is:

$$\begin{aligned}
\bar{R}_{ab} &= 2\bar{g}_{ab}\,, \\
R^{(1)}_{ab} &= \bar{g}_{ab}\left(-\frac{1}{2}(\nabla_\nu\nabla^\nu N + \nabla_b\nabla^b N) - \frac{1}{6}\nabla_b\nabla^b(3M+N)\right) - \frac{1}{2}(3M+N)_{;(ab)}\,, \\
R^{(2)}_{ab} &= \bar{g}_{ab}\frac{1}{12}(4(N^2)_{;c}^{\ \ c} + 6\Big(N_{,\mu}h^{\mu\nu})_{;\nu} + (h^{2\ \mu}_{\mu})_{;c}^{\ \ c} - h^{\mu\nu}_{\ \ ;c}h_{\mu\nu;}^{\ \ \ \ c} - N_{,c}N_{,}^{\ c}\Big) + \\
&\quad + \frac{1}{4}\Big((N^2 + h^{2\ \mu}_{\mu})_{;(ab)} - h^{\mu\nu}_{\ \ ;(a}h_{\mu\nu;b)} - N_{,(a}N_{,b)}\Big)\,.
\end{aligned} \tag{A.2}$$

We should also compute (A.1) for the case $M = \mu$, $N = a$, but because the only thing we extract from this is h', needed only in linearized approximation we refer to [16].

Next we consider the forms and the scalars given in (3.2) and we need only the fluctuations in Hodge duality for forms and in the covariant derivative for scalars, due to metric fluctuations:

$$
\begin{aligned}
*H_{MNP} &= \frac{1}{3!}\epsilon_{MNP}{}^{QRS}\left[\left(1+\frac{1}{2}h_M{}^M+\frac{1}{4}\left(\frac{1}{2}(h_M{}^M)^2+h^2{}_M{}^M\right)\right)H_{QRS}+\right.\\
&\qquad\left. +3\left(-h_Q{}^{Q'}+h^2{}_Q{}^{Q'}\right)H_{Q'RS}+6h_Q{}^{Q'}h_R{}^{R'}H_{Q'R'S}\right]+\cdots,\\
D_M P^{ir}_N &= \frac{1}{\sqrt{2}}\left(\bar{\nabla}_M\partial_N\phi^{ir}+\Gamma^{(1)}_{MN}{}^P\partial_P\phi^{ir}\right)+\cdots,
\end{aligned}
\tag{A.3}
$$

Using the definitions for fields, we obtain in a straightforward manner the equations given in text.

B. Spherical harmonics on S^3

We list in this appendix the formulas we need to know about spherical harmonics on S^3. We define the scalar harmonics similar to [3], namely:

$$
Y^I \equiv \frac{1}{r^l}x^{(i_1}\cdots x^{i_l)}\Omega^I_{i_1\cdots i_l}\,,
\tag{B.1}
$$

where x^i, $i=1,\ldots,4$ are four flat coordinates, $r=\sqrt{{x_1}^2+\cdots+{x_4}^2}$ and the sphere S^3 is embedded as $r=1$. The pharantheses $(\cdots)$ means like anywhere in text, the symmetrized and traceless tensor and Ω^I's form an orthonormal basis of constant symmetrized and traceless tensors. From (B.1) is clear that Y^I does not depend on r, but only on sphere indices. We also need:

$$
\begin{aligned}
\nabla_b\nabla^b Y^I &= -l(l+2)Y^I\,,\\
\nabla_b\nabla^b(\partial_a Y^I) &= (-l(l+2)+2)\partial_a Y^I\,.
\end{aligned}
\tag{B.2}
$$

We define $\Delta=l(l+2)$. Considering the basic constants we need to compute and determine all the other spherical harmonic related constants in term of these:

$$
\begin{aligned}
C(I) &\equiv \int_{S^3} Y^I Y^I\,,\\
C(I,I_1,I_2) &\equiv \int_{S^3} Y^I Y^{I_1} Y^{I_2}\,.
\end{aligned}
\tag{B.3}
$$

We can now prove using eventually simple insertions of $\nabla_b\nabla^b$ in the equations (B.3) the following identities:

$$
\begin{aligned}
\int_{S^3} Y^I Y^{I_1} &= C(I)\delta^{I,I_1}\,,\\
\int_{S^3} \partial_a Y^I\partial^a Y^{I_1} &= C(I)\delta^{I,I_1}\Delta\,,\\
\int_{S^3} \nabla_{(a}\nabla_{b)}Y^I\nabla^{(a}\nabla^{b)}Y^{I_1} &= C(I)\delta^{I,I_1}\frac{2}{3}\Delta(\Delta-3)
\end{aligned}
\tag{B.4}
$$

and

$$
\begin{aligned}
\int_{S^3} Y^I \nabla_a Y^{I_1} \nabla^a Y^{I_2} &= \frac{1}{2}(\Delta_1+\Delta_2-\Delta) C(I,I_1,I_2)\,, \\
\int_{S^3} \nabla^{(a}\nabla^{b)} Y^I \nabla_{(a} Y^{I_1} \nabla_{b)} Y^{I_2} &= \frac{1}{4}(\Delta_1+\Delta_2-\Delta-4)(\Delta_1+\Delta_2-\Delta) C(I,I_1,I_2)\,, \\
\int_{S^3} Y^I \nabla_{(a}\nabla_{b)} Y^I \nabla^{(a}\nabla^{b)} Y^{I_1} &= \left(\frac{1}{4}(\Delta_1+\Delta_2-\Delta-4)(\Delta_1+\Delta_2-\Delta)-\frac{1}{3}\Delta_1\Delta_2\right)\times \\
&\quad\times C(I,I_1,I_2)\,. \qquad \text{(B.5)}
\end{aligned}
$$

We compute the constants defined in (B.3) using gaussian integrals in x-space and we obtain the following:

$$
\begin{aligned}
C(I) &= \frac{2\pi^2}{2^l(l+1)}\,, \\
C(I,I_1,I_2) &= \frac{2\pi^2}{2^{\Sigma/2}} \frac{l!\,l_1!\,l_2!}{(\alpha/2)!\,(\alpha_1/2)!\,(\alpha_2/2)!\,(\Sigma/2+1)!} \langle \Omega \Omega^{I_1} \Omega^{I_2} \rangle\,, \qquad \text{(B.6)}
\end{aligned}
$$

where the last factor represents the possible contractions of all the indices of the three normalized tensors.

After working everything out we write a $D=3$ action responsible for the reduced (on S^3) equations of motion. It is the spirit of the correspondence to relate the higher dimensional vertices to correlation functions in a conformal field theory one lower dimension. The explicit correspondence was given in [2] and we repeat here the results applicable to our case. The two and three point correlation functions in a conformal field theory are completely determined up to a constant. We give below only the constants assuming that the linearized equations are those of correponding chirals (η is a constant coming from higher dimensional supergravity and $\delta = l(l-2)$):

$$
\begin{aligned}
\frac{\eta}{2}((\partial s)^2 + \delta\, s^2) &\longrightarrow \eta \frac{l(l-1)}{\pi}\,, \\
\eta\, s s_1 s_2 &\longrightarrow \eta \frac{(\alpha/2-1)!\,(\alpha_1/2-1)!\,(\alpha_2/2-1)!\,(\Sigma/2-1)!}{2\pi^2 (l-2)!\,(l_1-2)!\,(l_2-2)!}\,. \qquad \text{(B.7)}
\end{aligned}
$$

References

[1] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Correlation functions in the CFT_d/AdS_{d+1} correspondence*, *Nucl. Phys.* **B 546** (1999) 96 [hep-th/9804058].

[3] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, *Three-point functions of chiral operators in $D=4$, $N=4$ SYM at large N*, *Adv. Theor. Math. Phys.* **2** (1998) 697 [hep-th/9806074].

[4] S. Lee, *AdS_5/CFT_4 four-point functions of chiral primary operators: cubic vertices*, *Nucl. Phys.* **B 563** (1999) 349 [hep-th/9907108].

[5] G. Arutyunov and S. Frolov, *Some cubic couplings in type-IIB supergravity on $AdS_5 \times S^5$ and three-point functions in SYM_4 at large-N*, hep-th/9907085.

[6] J. Maldacena and A. Strominger, *AdS_3 black holes and a stringy exclusion principle*, *J. High Energy Phys.* **12** (1998) 005 [hep-th/9804085].

[7] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B 379** (1996) 99 [hep-th/9601029].

[8] J. de Boer, *large-N elliptic genus and AdS/CFT correspondence*, *J. High Energy Phys.* **05** (1999) 017 [hep-th/9812240].

[9] F. Larsen and E. Martinec, U(1) *charges and moduli in the D1-D5 system*, *J. High Energy Phys.* **06** (1999) 019 [hep-th/9905064].

[10] N. Seiberg and E. Witten, *The D1/D5 system and singular CFT*, *J. High Energy Phys.* **04** (1999) 017 [hep-th/9903224].

[11] A. Jevicki and S. Ramgoolam, *Non-commutative gravity from the AdS/CFT correspondence*, *J. High Energy Phys.* **04** (1999) 032 [hep-th/9902059].

[12] A. Jevicki, M. Mihailescu and S. Ramgoolam, *Gravity from CFT on $S^N(X)$: symmetries and interactions*, hep-th/9907144.

[13] R. Corrado, B. Florea and R. McNees, *Correlation functions of operators and Wilson surfaces in the $D = 6$, $(0,2)$-theory in the large-N limit*, *Phys. Rev.* **D 60** (1999) 085011 [hep-th/9902153].

[14] F. Bastianelli and R. Zucchini, *Three point functions of chiral primary operators in $D = 3$, $N = 8$ and $D = 6$, $N = (2,0)$ SCFT at large N*, *Phys. Lett.* **B 467** (1999) 61 [hep-th/9907047].

[15] L.J. Romans, *Selfduality for interacting fields: covariant field equations for six-dimensional chiral supergravities, Nucl. Phys.* **B 276** (1986) 71.

[16] S. Deger, A. Kaya, E. Sezgin and P. Sundell, *Spectrum of $D = 6$, $N = 4B$ supergravity on $AdS_3 \times S_3$*, *Nucl. Phys.* **B 536** (1998) 110 [hep-th/9804166].

[17] H. Nastase, D. Vaman and P. van Nieuwenhuizen, *Consistent nonlinear KK reduction of 11d supergravity on $AdS_7 \times S_4$ and self-duality in odd dimensions*, *Phys. Lett.* **B 469** (1999) 96 [hep-th/9905075].